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ON A PROBLEM IN THE THEORY OF THE UNIDIRECTIONAL REGENERATOR

ALAN KARDAS
Institute of Gas Technology, Chicago

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Abstract—An analytical solution of the unidirectional regenerator problem is presented. The effectiveness of a regenerator at any arbitrary distance from the inlet is defined in terms of dimensionless groups determining the problem, computed for a wide range of parameters, and given in a series of graphs. While retaining the limitation of equal heat capacities and constant heat-transfer coefficients for hot and cold gas streams, the analytical treatment differs from the conventional methods of attack in that:

(1) No simplifying assumptions are made with regard to the material properties and dimensions of the heat storage matrix other than postulating that the thermal conductivity of the solid be zero in the direction of gas flow.
(2) No assumptions are needed regarding the nature of the longitudinal temperature profile along the regenerator.

Local effectiveness graphs, which describe the operation of a unidirectional regenerator in cyclic operation, are of the type heretofore available only for ordinary recuperators.

NOMENCLATURE

A, integration constant [equations (12), (13), (15)]; heat-transfer area [equations (24) to (26)];
a, halfthickness of plate [ft];
B_j, a convenience parameter, defined in equation (26);
b, halfdistance between two plates [ft];
C, amplitude of T, defined by equation (19a);
c'_p, heat capacity of fluid [Btu/(ft^3 degF)];
c_p, heat capacity of fluid [Btu/(lb degF)];
e, base of natural logarithms, also written exp;
f_c, f_s, f_c^*, f_s^*, frequently recurring groups defined in terms of hyperbolic functions and trigonometric functions immediately following equation (19a);
G, mass velocity of fluid [lb/(h ft^2)];
g_c, g_s, groups similar to the f_c, f_s described above, defined *ibidem*;
h, surface (film) heat-transfer coefficient in Newtonian heating or cooling [Btu/(h ft^2 degF)];
i, imaginary unit, $i = \sqrt{(-1)}$;
i, j, subscripts denoting sequences of integers 1, 2, 3, . . . ;
k, thermal conductivity of plate [Btu/(h ft degF)];
M, frequency parameter, dimensionless, $M = \sqrt{(\omega'/2)}$;
M^*, same, defined in terms of M: $M^* = M\xi$;
N_{Bi}, Biot number, dimensionless, $N_{Bi} = ha/k$;
q, a transform variable, $s = q^2$;
Q, heat energy [Btu];
$Q_{\tau_0/2}$, quantity of energy absorbed or given up by the plate during half cycle [Btu];
R, a convenience parameter, $R = (1 + \eta)M/N_{Bi}$;
r, radius of a contour of integration in the complex plane s;
S, a convenience parameter, defined in equation (20a);
s, complex frequency (Laplace transform variable);

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T, temperature ratio, t/t'_g;
T_g, fluid temperature [°F];
T_h, temperature of incoming hot stream [°F];
T_c, temperature of incoming cold stream [°F];
t, plate temperature referred to mean base temperature zero [degF];
t_g, fluid temperature referred to mean base temperature zero [degF];
t'_g, maximum amplitude of fluid temperature oscillation at entry to regenerator. It is equal to one half of the temperature difference between the incoming streams [degF];
t_s, same as t, only at the surface of plate;
U, a convenience parameter, defined in equation (20a);
v, fluid velocity [ft/h];
x, distance along the direction of flow [ft];
y, distance along the direction normal to the direction of flow [ft].

Greek symbols
α, thermal diffusivity of the plate [ft^2/h];
β_i, a root of the equation $\beta \tan \beta = N_{Bi}/(1 + \eta)$;
γ, a finite real number greater than zero;
ϵ^*, phase parameter, defined in equation (19a);
ϵ, ϵ^* evaluated at $\xi = 1$;
$\epsilon_{g,j}$, phase parameter, defined in equation (26);
ζ, "non-dimensional" time, $\zeta = \alpha(v\tau - x)/va^2$;
η, "non-dimensional" distance along the plate, $\eta = xh/(c_p b \rho v)$;
θ, temperature ratio t_g/t'_g;
ξ, thickness or depth ratio, $\xi = y/a$;
ρ, fluid density [lb/ft^3];
τ, time [h];
τ_0, length of total cycle [h];
Ψ, effectiveness, a dimensionless ratio. Its definition follows the statement of equation (26);
ω, frequency [h^{-1}];
ω' modified frequency, dimensionless, $\omega' = \omega a^2/\alpha$.

$\bar{f}$, the bar over a letter denotes a Laplace-transformed function. Thus $\bar{f}(s) = \int_0^\infty \exp[-s\zeta] f(\zeta)\, d\zeta$ for any function f for which this operation is defined.

1. INTRODUCTION

ALTHOUGH the principle of regenerative heat exchange is quite common, and regenerators have been built for many years, the underlying theory and accurate mathematical descriptions did not appear until the second and third decades of this century.

In its earliest stages, the theory was highly approximative and utilitarian. Two main trends may be discerned in its development: on the one hand, in the industry concerned with the construction of high-temperature air preheaters, the temperature distribution in the direction of flow was usually assumed to be the same as in a recuperator operating under similar conditions, while the temperature distribution in the brick normal to the direction of flow was studied by postulating mean temperatures or making plausible assumptions about the nature of such temperature profiles. The resulting solutions were semi-empirical, with constants taken from the wealth of experimental data obtained in building and operating blast furnace stoves and air preheaters for similar industrial applications. A good description of this approach is given by A. Schack [1].

On the other hand, the designers of low-temperature regenerators to be used in liquefaction of air who worked with heat storage matrices made of thin strips of aluminum or steel postulated no thermal gradients in the material. They studied the temperature distribution along the regenerator without first assuming that it would be similar to that found in a comparable recuperator.

Attempts were also made to obtain a general mathematical description of regenerative heat transfer without assuming infinite conductivity of the matrix or prescribing a certain mean temperature distribution along the regenerator, but the absence of effective computing machinery made the solution of such descriptions impracticable. Thus Schmeidler [2] wrote down a set of integral equations for the cyclic steady state of a regenerator and indicated an approximative solution in terms of mean temperatures. Ackermann [3] presented a detailed theory of regenerators which, however, yields a solution for a given case only after repeated calculations which lead to a cyclic steady state. Lowan [4] offered a solution for a regenerator with cylindrical heat storage matrix. His solution requires repeated summing of several series in terms of Bessel functions of imaginary argument. The widely used textbook by Jakob [5] presents the set of partial differential equations and boundary conditions describing the problem in question with the remark that the analytical solution is difficult and with a brief reference to the work of Ackermann cited earlier.

A comprehensive bibliography of the various attempts at solution of the regenerator problem is given by Hausen [6] (scattered in footnotes throughout the text).

Finally, to conclude this introduction, a word about the relation of the present work to Hausen's standard treatise on the subject. Hausen, whose exhaustive monograph [6] on recuperators and regenerators covers the state of the theory up to 1950, bases his method of solution on decomposing the cyclic temperature behavior of the gas into a base oscillation and the higher harmonics. The main oscillation is again assumed to correspond to the temperature distribution in a recuperator, while the higher harmonics describe the behavior of a purely regenerative nature. The method is used to treat both the unidirectional and the countercurrent cases with many refinements of great practical significance. However, the fundamental assumption that the main oscillation of a regenerator is identical with the temperature distribution along a recuperator imposes the requirement that time-mean temperatures be used, and special overall heat-transfer coefficients have to be defined. For actual calculations Hausen introduces a "heat-pole" method which entails considerable computational or graphical effort and which, therefore, is offered on several levels of complexity (and precision).

By comparison, the solution offered below treats only the unidirectional case with the usual assumption of zero thermal conductivity in the solid matrix in the direction of flow, and retaining the limitation of equal heat capacities and constant heat-transfer coefficients for hot and cold gas streams. No other restrictions are placed on the thermal properties of the heat storage material or on its dimensions. The solid matrix is taken to consist of an array of parallel plates of finite thickness and length. The method of solution, however, could also be applied, with appropriate modifications, to a matrix consisting of cylindrical rods disposed in the direction of the fluid flow or to a matrix made up of spheres of finite thermal conductivity. The thermal properties of the fluid, its velocity, duration of the cycle, and regenerator length are arbitrary.

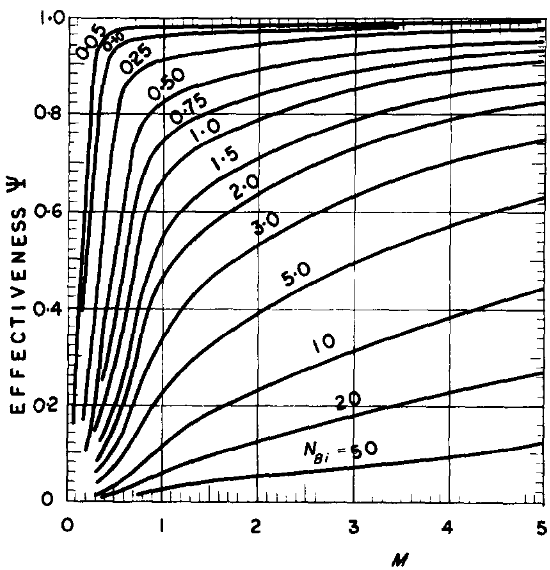

FIG. 1. Regenerator effectiveness, $\eta = 0$.

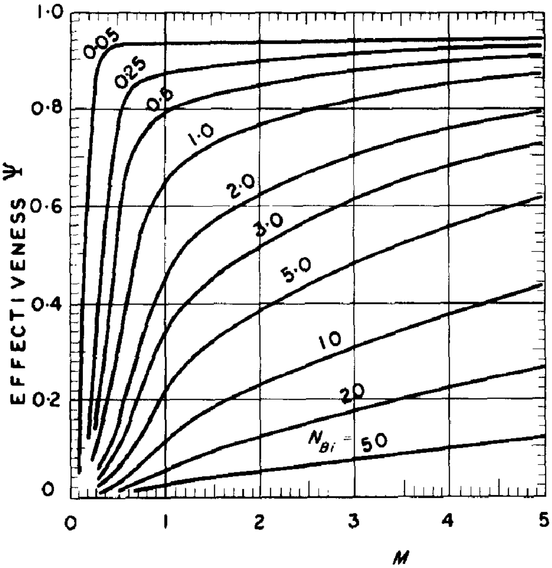

FIG. 2. Regenerator effectiveness, $\eta = 0{\cdot}05$.

Because of the complexity of the final solution, and in order to make the results readily accessible for possible use, the solution was carried out in terms of convenient non-dimensional parameters, calculated on an electronic computer for a wide range of these parameters, and prestented in graphical form on Figs. 1–6.

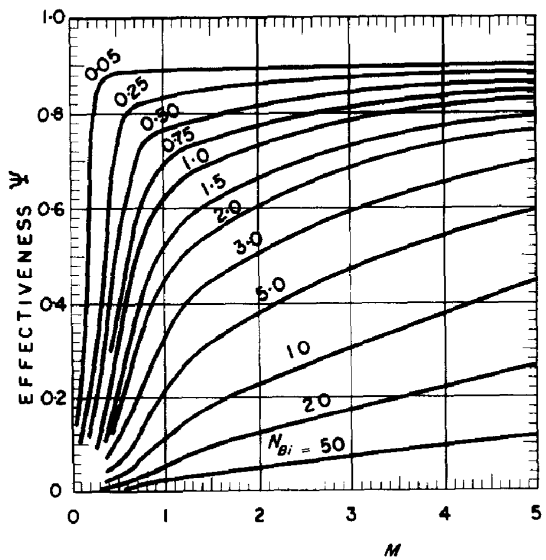

FIG. 3. Regenerator effectiveness, $\eta = 0{\cdot}1$.

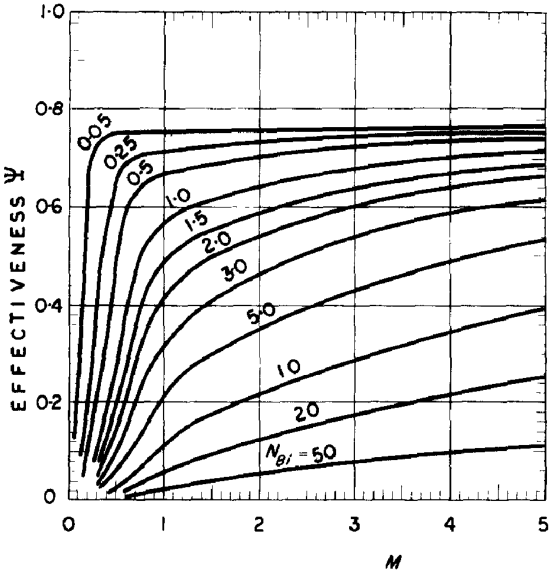

FIG. 4. Regenerator effectiveness, $\eta = 0{\cdot}3$.

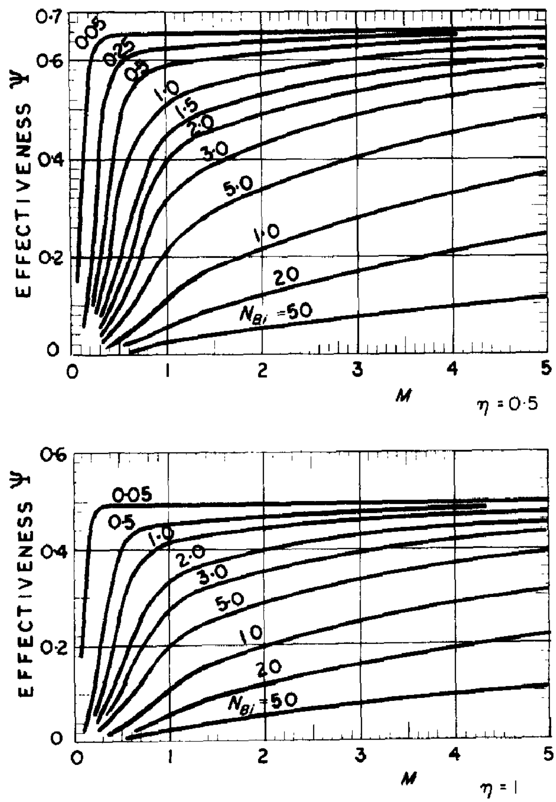

FIG. 5. Regenerator effectiveness.

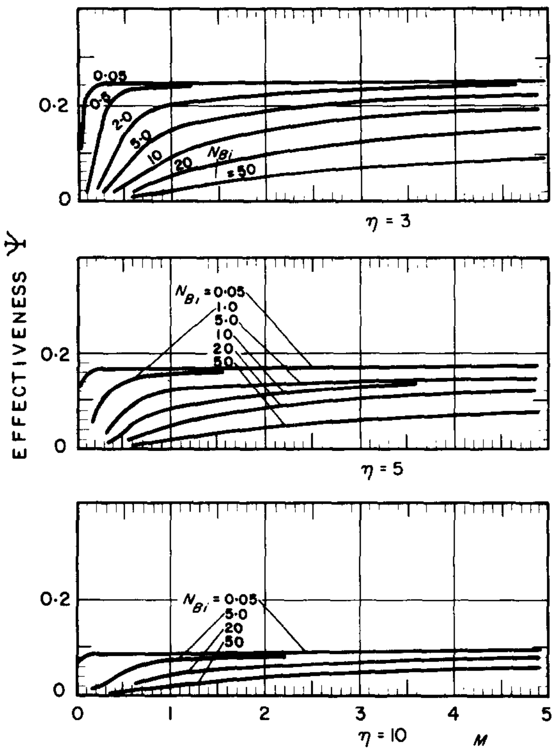

FIG. 6. Regenerator effectiveness.

2. DISCUSSION OF THE PROBLEM AND SOLUTION

Consider an array of parallel plates, each plate being $2a$ units thick and separated from the next plate by the distance of $2b$ units of length. Because of symmetry, it is sufficient to consider one half of a plate extending in a Cartesian coordinate system from $x = 0$ to any desired length in the direction of positive x, and from $y = 0$ at the center of the plate to $y = a$ at the surface of the plate.

The plate is in contact with a well mixed layer of fluid of thickness $2b$ of which, again because of symmetry, only one half, b units thick, is considered. The fluid enters at $x = 0$ at a temperature $\exp[-i\omega\tau]$ normalized about the average incoming temperature, and at a constant mass flow rate, moving in the direction of positive x. As it proceeds along the plate, it exchanges the heat energy with the plate according to Newton's [7] law of cooling and heating of bodies in air. The plate is assumed to have zero conductivity in the x-direction and finite conductivity, k, in the y-direction.

The foregoing verbal description may be rendered analytically in dimensionless form as follows: in the plate

$$\frac{\partial T}{\partial \zeta} = \frac{\partial^2 T}{\partial \xi^2} \tag{1}$$

and in the fluid

$$\frac{\partial \theta}{\partial \eta} = -(\theta - T) \tag{2}$$

with the boundary conditions

$$\frac{\partial T}{\partial \xi} = N_{Bi}(\theta - T), \qquad \xi = 1 \tag{3}$$

$$\theta = \exp[-i\omega'\zeta], \qquad \eta = 0 \tag{4}$$

$$\frac{\partial T}{\partial \xi} = 0, \qquad \xi = 0 \tag{5}$$

$$T = 0, \qquad \zeta = 0. \tag{6}$$

Equations (1) and (2) may now be solved, subject to conditions (3) through (6), using the Laplace transform. Let

$$\bar{f}(s) = \int_0^\infty \exp[-s\zeta] f(\zeta)\, \mathrm{d}\zeta.$$

Then equations (1) through (5) are, in terms of the transform variables,

$$\frac{\mathrm{d}^2 \bar{T}}{\mathrm{d}\xi^2} = s\bar{T} \tag{7}$$

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\eta} = -(\bar{\theta} - \bar{T}), \qquad \xi = 1 \tag{8}$$

$$\frac{\mathrm{d}\bar{T}}{\mathrm{d}\xi} = N_{Bi}(\bar{\theta} - \bar{T}), \qquad \xi = 1 \tag{9}$$

$$\bar{\theta} = \frac{1}{s + i\omega}, \qquad \eta = 0 \tag{10}$$

$$\frac{\mathrm{d}\bar{T}}{\mathrm{d}\xi} = 0, \qquad \xi = 0. \tag{11}$$

Condition (6) has been incorporated into equation (7).

A solution of equations (7) and (11) is, for any η, with $s = q^2$,

$$\bar{T} = A \cosh q\xi. \tag{12}$$

Condition (9) serves to determine A in equation (12):

$$\left.\frac{\partial \bar{T}}{\partial \xi}\right|_{\xi=1} = Aq \sinh q = N_{Bi}(\bar{\theta} - A \cosh q),$$

whence

$$A = \frac{\bar{\theta}}{(q/N_{Bi}) \sinh q + \cosh q} \tag{13}$$

If, now, the value of $\bar{\theta}$ from equation (13) is substituted into a solution of (8), and the condition (10) is applied, the result is

$$\bar{\theta} = -\frac{Aq \sinh q}{N_{Bi}} \eta + \frac{1}{s + i\omega'} \tag{14}$$

Elimination of $\bar{\theta}$ from equations (13) and (14) leads to

$$A = \frac{1}{(s + i\omega')\{[(1 + \eta)/N_{Bi}]\, q \sinh q + \cosh q\}} \tag{15}$$

and thus to solutions of equations (7) and (8).

$$\bar{T} = A \cosh q\xi \tag{16}$$

$$\bar{\theta} = \frac{1}{s + i\omega'} - \frac{A\eta}{N_{Bi}} q \sinh q \tag{17}$$

where A is given by equation (15).

The return from the complex frequency plane s to the domain of the variable ζ is effected by means of the inversion integral,

$$T = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\exp[s\zeta] \cosh q\xi \, ds}{(s + i\omega')\{[(1 + \eta)/N_{Bi}]\, q \sinh q + \cosh q\}} \tag{18}$$

which may be evaluated using residue theory. In particular, the only residue of interest is that due to the simple pole $s = -i\omega'$ in the complex s-plane. The residues at the poles of the expression $\{[(1 + \eta)/N_{Bi}]\, q \sinh q + \cosh q\}^{-1}$ contribute only to the transient part of the solution, which is of limited interest in actual operation of a regenerator and will not be discussed here. It is easily seen that they contribute nothing to the cyclic steady-state part of the solution. Indeed, setting the expression inside the square brackets equal to zero, one obtains the transcendental equation $q \tanh q = -N_{Bi}/(1 + \eta)$ or, for a complex $q = i\beta$, $\beta \tan \beta = N_{Bi}/(1 + \eta)$. The roots of this equation, written in terms of $s = q^2$, i.e. $-\beta_i^2$ for $i = 1, 2, 3, \ldots$, provide the damping factor, $\exp[-\beta_i^2 \zeta]$, which tends to zero as steady state is approached and ζ grows large.

The simple pole $s = -i\omega'$ lies inside the contour delimited by the straight line $s = \gamma$, where γ is any real positive number, and that portion of the circle of radius r, ($r \to \infty$) with the origin as center, which is to the left of this line. The theory of residues [8] applies, with the result that

$$\begin{aligned} T &= \lim_{s \to -i\omega'} \frac{\exp[s\zeta] \cosh \sqrt{(s\xi)}}{(s + i\omega')\{[(1 + \eta)/N_{Bi}](\sqrt{s}) \sinh(\sqrt{s}) + \cosh(\sqrt{s})\}} \\ &= \frac{\exp[-i\omega'\zeta] \cosh[\sqrt{(\omega'/2)}(i - 1)\xi]}{[(1 + \eta)/N_{Bi}](i - 1)\sqrt{(\omega'/2)} \sinh[\sqrt{(\omega'/2)}(i - 1)] + \cosh[(i - 1)\sqrt{(\omega'/2)}]} \end{aligned} \tag{19}$$

where $\sqrt{(-\omega')} = (i - 1)\sqrt{(\omega'/2)}$.

Applying the inversion integral to equation (17) and using the theory of residues provides the value of

$$\theta = \exp[-i\omega'\zeta]$$
$$\left[1 - \frac{\eta(i-1)\sqrt{(\omega'/2)}\sinh[(i-1)\sqrt{(\omega'/2)}]}{(1+\eta)(i-1)\sqrt{(\omega'/2)}\sinh[(i-1)\sqrt{(\omega'/2)}] + N_{Bi}\cosh[(i-1)\sqrt{(\omega'/2)}]}\right] \quad (20)$$

After a series of elementary transformations and simplifications, which have been relegated to Appendix 1, and upon taking the real parts of the final results, equations (19) and (20) become, respectively, (19a) and (20a):

$$T = C\cos(\omega'\zeta + \epsilon^*) \quad (19a)$$

where

$$C = \left[\frac{\cosh^2 M^* - \sin^2 M^*}{2R^2(\sinh^2 M + \sin^2 M) + 2R[f_c(g_s - g_c) + f_s(g_s + g_c)] + \cosh^2 M - \sin^2 M}\right]^{\frac{1}{2}}$$

$$\epsilon^* = \arctan\frac{f_s^*[R(g_s - g_c) + f_c] - f_c^*[R(g_s + g_c) + f_s]}{f_c^*[R(g_s - g_c) + f_c] + f_s^*[R(g_s + g_c) + f_s]}$$

and

$$R = (1+\eta)M/N_{Bi};$$
$$M = \sqrt{(\omega'/2)} = \sqrt{(\omega a^2/2\alpha)} = \sqrt{(\pi a^2/\alpha\tau_0)};$$
$$M^* = \xi\sqrt{(\omega'/2)} = \sqrt{(\omega y^2/2\alpha)};$$

$$f_c = \cosh M\cos M$$
$$f_s = \sinh M\sin M$$
$$g_c = \cosh M\sin M$$
$$g_s = \sinh M\cos M$$
$$f_c^* = \cosh M^*\cos M^*$$
$$f_s^* = \sinh M^*\sin M^*$$

$$\theta = \sqrt{(S^2 + U^2)}\cos(\omega'\zeta + \arctan U/S) \quad (20a)$$

where

$$U = \frac{-\eta M(\sinh M\cosh M + \sin M\cos M)}{N_{Bi}[(1+2R^2)\sinh^2 M - (1-2R^2)\sin^2 M + 1 + 2R(\sinh M\cosh M - \sin M\cos M)]}$$

$$S = 1 + U\frac{2R(\sinh^2 M + \sin^2 M) + (\sinh M\cosh M - \sin M\cos M)}{\sinh M\cosh M + \sin M\cos M}$$

Equation (19a) describes the thermal response anywhere in a plate exposed to a fluid which enters at $x = 0$ at the temperature $\exp[-i\omega\tau]$ and moves along the plate. The fluid, in turn, behaves according to equation (20a) as it exchanges its energy with the plate in its progress in the direction of the increasing x. Taken together, these two equations describe the energy transfer from the hot fluid to the cold fluid in a cyclic steady operation of a unidirectional regenerator. It is difficult, however, to envisage a regenerator "driven" by a fluid which enters the heat storage matrix with the time–temperature behavior of a perfect sine wave. A regenerator is more likely to be operated by alternating the incoming streams in an on–off manner of a square wave. A solution corresponding to such input is obtained by performing a harmonic analysis of the desired arbitrary fluctuation, calculating the temperatures T and θ for each harmonic of the resulting series, and summing.

In the subsequent analysis the square waveform will be taken to extend from +1 (entering hot stream, first half of the cycle) to −1 (entering cold stream, second half of the same cycle). The actual fluid temperature, T_g, is then recovered, provided the temperature of the incoming hot stream, T_h, the temperature of the incoming cold stream, T_c, and the solution θ are known:

$$\theta = \frac{2T_g - T_h - T_c}{T_h - T_c} \tag{21}$$

Decomposing the square wave in the usual manner (Fourier [9], p. 143) and summing the harmonic components one obtains the plate temperature

$$T = \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{C_j}{2j-1}\sin(\omega'_j\zeta + \epsilon_j) \tag{22}$$

and the fluid temperature

$$\theta = \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{\sqrt{(S_j^2 + U_j^2)}}{2j-1}\sin\left(\omega'_j\zeta + \arctan\frac{U_j}{S_j}\right) \tag{23}$$

with the coefficients C_j, S_j, and U_j written in terms of the component parameters $M_j = \sqrt{[(2j-1)\omega'/2]} = \sqrt{(\omega'_j/2)}$ and $R_j = (1+\eta)M_j/N_{Bi}$.

It should be noted that equations (22) and (23) could have been obtained in a more direct manner by writing the Laplace transform of a square wave† into the right-hand side of equation (10) and repeating the development which led to equations (19a) and (20a). However, the method followed here of obtaining the response to a harmonic excitation first and synthesizing afterwards affords greater flexibility. It presents the building blocks from which a solution to a problem involving an arbitrary forcing function may be constructed with comparative ease.

3. INTERPRETING THE SOLUTIONS AND USING THEM

Although equations (22) and (23) may be used to gain some insight into the interrelated temperature fields of the gas and the solid and their dependence on time and space, their importance to the designer of a regenerative system is secondary. Of more interest to the designer is the problem of effective utilization of energy and of optimum use of the heat storage material. In order to provide an answer to this problem, it is necessary to calculate the quantity of heat alternately stored in and delivered by the heat storage matrix at any given point along the regenerator. This may be done by calculating the heat transferred from the gas to the surface of the plate:

$$dQ = hA(\theta - T)t'_g\, d\tau \tag{24}$$

where dQ is the quantity of heat transferred during the time interval $d\tau$, and T is evaluated at the surface of the plate. Integrating over a half cycle, we obtain

$$Q_{\tau_0/2} = hAt'_g\left(\int_0^{\tau_0/2} \theta\, d\tau - \int_0^{\tau_0/2} T\, d\tau\right) \tag{25}$$

Integration over the total cycle would serve no useful purpose since a cyclic steady state is presumed to exist and the integral of dQ for the whole period is equal to zero. On performing the operations indicated in equation (25) on the expressions for θ and T given in equations (22) and (23) and simpli-

† $\frac{1}{s}\tanh\frac{s\tau_0}{4}$.

fying† the following expression for the quantity of heat entering or leaving the plate at any position x during a half period is obtained:

$$Q_{\tau_0/2} = \frac{hA\tau_0 t'_g 8}{2\pi^2} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} \left\{ B_j \cos\left[\frac{(2j-1)\pi 2x}{\tau_0 v} + \epsilon_{g,j}\right] - C_j \cos\left[\frac{(2j-1)\pi 2x}{\tau_0 v} - \epsilon_j\right]\right\}. \quad (26)$$

where

$$B_j = \sqrt{(S^2 + U^2)}$$

and

$$\epsilon_{g,j} = \arctan (U_j/S_j).$$

The group $hAt'_g\tau_0/2$ represents an ideal amount of heat which would be exchanged if the temperature of cold stream could be raised to the entrance temperature of the hot stream. The remaining factor in equation (26) is the local effectiveness of any position in a unidirectional regenerator. It will be denoted by the Greek letter Ψ. Thus, equation (26) becomes

$$Q_{\tau_0/2} = \frac{hAt'_g\tau_0}{2}\Psi \quad (26a)$$

The use of a local effectiveness Ψ as a measure of the operational efficiency of a regenerator represents a departure from the more general definition (e.g. Jakob [5], p. 268) which sought to relate the efficiencies of regenerators and recuperators. The purpose of the present paper is more direct: to provide the designer of a unidirectional regenerator with the effectiveness of the system at any dimensionless length η. The results are given on Figs. 1–6, where Ψ is plotted against the frequency parameter M for a variety of Biot numbers. Each figure represents a different value of the distance parameter $xh/c_pb\rho v$ in an increasing sequence from 0 to 10. The sequence is sufficiently dense, so that linear interpolation from figure to figure is possible. The results may be read with an accuracy of one or two per cent which is thought to be sufficient in the light of our imperfect knowledge of the thermal properties of the gas stream, of the solid matrix, and of the film coefficient h.

The ratio x/v which is used in the formulation of the distance parameter $xh/c_pb\rho v$ also appears in the definition of dimensionless time, and in the arguments of the trigonometric functions in equation (26), where it is divided by $\tau_0/2$. Although it represents but a negligible fraction of the total period on the wide range covered by Figs. 1–6, it was carried in the development and retained in equation (26). In numerical applications, several values of the ratio $x/(v\tau_0)$ were tried and the results compared. At high frequency numbers ($M = 5$), the values of Ψ were found to be lower by less than 0·005 for $x/(v\tau_0) = 0{\cdot}01$ than the corresponding results for $x/(v\tau_0) = 0{\cdot}001$. The agreement at low frequency numbers was within the computational accuracy. Figures 1–6 were plotted from the results computed for ratios $x/(v\tau_0) = 0{\cdot}0001$ and $x/(v\tau_0) = 0{\cdot}001$, but the scale is such that the effect of the parameter $x/(v\tau_0)$ cannot be seen in any case.

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† For details, see Appendix 2.

APPENDIX 1

Reduction of Complex Solutions (19) *and* (20) *to their Real Parts* [*Equations* (19a) *and* (20a)]

Equation (19) was written:

$$T = \frac{\exp[-i\omega'\zeta]\cosh[\sqrt{(\omega'/2)}(i-1)\xi]}{[(1+\eta)/N_{Bi}](i-1)\sqrt{(\omega'/2)}\sinh[\sqrt{(\omega'/2)}(i-1)] + \cosh[\sqrt{(\omega'/2)}(i-1)]}$$

Using the identities

$$\sinh(a - ib) = \sinh a \cos b - i\cosh a \sin b$$

$$\cosh(a - ib) = \cosh a \cos b - i \sinh a \sin b$$

and the variables f_c, f_s, g_c, g_s defined in the text immediately following equation (19a), equation (19) may be written

$$T = \frac{\exp[-i\omega'\zeta](f_c^* - if_s^*)}{[(1+\eta)/N_{Bi}]M[(g_s - g_c) - i(g_s + g_c)] + f_c - if_s}$$

$$= \frac{\exp[-i\omega'\zeta](f_c^* - if_s^*)}{R(g_s - g_c) + f_c - i[R(g_s + g_c) + f_s]}.$$

Furthermore, let $E = R(g_s - g_c) + f_c$ and $F = R(g_s + g_c) + f_s$ and also $C^* = f_c^*E + f_s^*F$, $D^* = f_s^*E - f_c^*F$. Then

$$T = \frac{\exp[-i\omega'\zeta](C^* - iD^*)}{E^2 + F^2}.$$

From the definition of E and F and upon squaring,

$$E^2 + F^2 = 2R^2(\sinh^2 M + \sin^2 M) + 2R[f_c(g_s - g_c) + f_s(g_s + g_c)] + \cosh^2 M - \sin^2 M$$

while the real part of the numerator is $C^* \cos \omega'\zeta - D^* \sin \omega'\zeta$.

An application of the trigonometric identity

$$C^* \cos \omega'\zeta - D^* \sin \omega'\zeta = \sqrt{(C^{*2} + D^{*2})}\cos[\omega'\zeta + \tan^{-1}(D^*/C^*)]$$

leads to

$$T = \frac{\sqrt{(C^{*2} + D^{*2})}}{E^2 + F^2}\cos(\omega'\zeta + \epsilon^*).$$

But $C^{*2} + D^{*2} = (E^2 + F^2)(f_c^{*2} + f_s^{*2})$ from the definition of C^* and D^* and upon squaring. Therefore

$$T = \frac{\sqrt{[(E^2 + F^2)(f_c^{*2} + f_s^{*2})]}}{E^2 + F^2}\cos(\omega'\zeta + \epsilon^*)$$

$$= \sqrt{\left(\frac{\cosh^2 M\xi - \sin^2 M\xi}{E^2 + F^2}\right)}\cos(\omega'\zeta + \epsilon^*)$$

where $\epsilon^* = \tan^{-1}(D^*/C^*)$, and where the definition of f_c^*, f_s^* has been used. In order to obtain the final form of equation (19a), the variables E^2, F^2, C^* and D^*, which were introduced for convenience, are written out in full. This results in $T = C\cos(\omega'\zeta + \epsilon^*)$ with

$$C = \left[\frac{\cosh^2 M^* - \sin^2 M^*}{2R^2(\sinh^2 M + \sin^2 M) + 2R[f_c(g_s - g_c) + f_s(g_s + g_c)] + \cosh^2 M - \sin^2 M}\right]^{\frac{1}{2}}$$

and

$$\epsilon^* = \arctan\frac{f_c^*[R(g_s - g_c) + f_c] - f_c^*[R(g_s + g_c) + f_s]}{f_c^*[R(g_s - g_c) + f_c] + f_s^*]R(g_s + g_c) + f_s]}$$

which enters into equation (19a).

Equation (20) was written

$$\theta = \exp[-i\omega'\zeta]\left[1 - \frac{\eta(i-1)\sqrt{(\omega'/2)}\sinh[(i-1)\sqrt{(\omega'/2)}]}{(1+\eta)(i-1)\sqrt{(\omega'/2)}\sinh[(i-1)\sqrt{(\omega'/2)}] + N_{Bi}\cosh[(i-1)\sqrt{(\omega'/2)}]}\right].$$

Using the definitions and identities given earlier in this Appendix, θ is transformed into

$$\theta = \exp[-i\omega'\zeta]\left[1 - \frac{\eta M[(g_s - g_c) - i(g_s + g_c)]}{(1+\eta)M[(g_s - g_c) - i(g_s + g_c)] + N_{Bi}(f_c - if_s)}\right].$$

Let $g_s - g_c = J$, $g_s + g_c = K$, $(1+\eta)MJ + N_{Bi}f_c = X$, and $(1+\eta)MK + N_{Bi}f_s = Y$, so that

$$\theta = \exp[-i\omega'\zeta]\left[1 - \frac{\eta M(J - iK)}{X - iY}\right] = \exp[-i\omega'\zeta]\left[1 - \frac{\eta M[(JX + KY) + (JY - KX)i]}{X^2 + Y^2}\right]$$

and

$$\theta = \cos\omega'\zeta - i\sin\omega'\zeta - \frac{\eta M}{X^2 + Y^2}\{(\cos\omega'\zeta - i\sin\omega'\zeta)[JX + KY + i(JY - KX)]\}.$$

Taking only the real part of θ (which was its original definition as a certain temperature ratio), it becomes

$$\begin{aligned}\theta &= \cos\omega'\zeta - \frac{\eta M}{X^2 + Y^2}[(JX + KY)\cos\omega'\zeta + (JY - KX)\sin\omega'\zeta] \\ &= \left[1 - \frac{\eta M(JX + KY)}{X^2 + Y^2}\right]\cos\omega'\zeta - \frac{\eta M(JY - KX)}{X^2 + Y^2}\sin\omega'\zeta \\ &= S\cos\omega'\zeta - U\sin\omega'\zeta = \sqrt{(S^2 + U^2)}\cos[\omega'\zeta + \tan^{-1}(U/S)].\end{aligned}$$

The final expression is equation (20a). The convenience parameters written out in detail are

$$JX + KY = N_{Bi}[2R(\sinh^2 M + \sin^2 M) + (\sinh M\cosh M - \sin M\cos M)]$$

$$JY - KX = -N_{Bi}(\sinh M\cosh M + \sin M\cos M).$$

These, in turn, lead to the expressions for S and U:

$$S = 1 - \frac{\eta M[2R(\sinh^2 M + \sin^2 M) + (\sinh M\cosh M - \sin M\cos M)]}{N_{Bi}[(1 + 2R^2)\sinh^2 M - (1 - 2R^2)\sin^2 M + 1 + 2R(\sinh M\cosh M - \sin M\cos M)]}$$

$$U = -\frac{\eta M(\sinh M\cosh M + \sin M\cos M)}{N_{Bi}[(1 + 2R^2)\sinh^2 M - (1 - 2R^2)\sin^2 M + 1 + 2R(\sinh M\cosh M - \sin M\cos M)]}$$

which completes the development of equation (20a).

APPENDIX 2

Integration of the Energy Equation

The integration is to be performed on the known functions of time, the function θ as given in equation (23), and the function T given in equation (22) and evaluated at $y = a$, i.e. at $\xi = 1$. Before proceeding, dimensionless time ζ is converted to the real time, τ, and equation (25) becomes

$$Q_{\tau_0/2} = \frac{4hA}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j-1}\int_0^{\tau_0/2}\left\{B_j\sin\left[\omega_j\left(\tau-\frac{x}{v}\right)+\epsilon_{g,j}\right]-C_j\sin\left[\omega_j\left(\tau-\frac{x}{v}\right)+\epsilon_j\right]\right\}d\tau$$

$$= -\frac{4hA}{\pi}\sum_{j=1}^{\infty}\frac{1}{(2j-1)\omega_j}\left\{B_j\cos\left[\omega_j\left(\tau-\frac{x}{v}\right)+\epsilon_{g,j}\right]-C_j\cos\left[\omega_j\left(\tau-\frac{x}{v}\right)+\epsilon_j\right]\right\}_0^{\tau_0/2}$$

$$= -\frac{4hA\tau_0}{2\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\left\{B_j\cos\left[\omega_j\left(\frac{\tau_0}{2}-\frac{x}{v}\right)+\epsilon_{g,j}\right]-C_j\cos\left[\omega_j\left(\frac{\tau_0}{2}-\frac{x}{v}\right)+\epsilon_j\right]\right.$$
$$\left.-B_j\cos\left(\omega_j\frac{x}{v}-\epsilon_{g,j}\right)+C_j\cos\left(\omega_j\frac{x}{v}-\epsilon_j\right)\right\}$$

The coefficients B_j and C_j may be collected using the identity

$$\cos(m+n)-\cos n = -2\sin\frac{m+2n}{2}\sin\frac{m}{2},$$

leading to the following form of the expression for $Q_{\tau_0/2}$:

$$Q_{\tau_0/2} = \frac{4hA\tau_0}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\left\{B_j\sin\left[\frac{(2j-1)\pi}{2}-\frac{x(2j-1)2\pi}{v\tau_0}+\epsilon_{g,j}\right]\sin\frac{2j-1}{1}\right.$$
$$\left.-C_j\sin\left[\frac{(2j-1)\pi}{2}-\frac{x(2j-1)2\pi}{v\tau_0}+\epsilon_j\right]\cdot\sin\frac{(2j-1)\pi}{2}\right\}.$$

Further simplification is achieved by noting that $\sin[(2j-1)\pi/2+\epsilon_j] = -[(-1)^{j+1}]\cos\epsilon_j$. This leads directly to equation (26):

$$Q_{\tau_0/2} = \frac{hA\tau_0t_g'}{2}\frac{8}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\left\{B_j\cos\left[\frac{(2j-1)\pi}{\tau_0}\frac{2x}{v}+\epsilon_{g,j}\right]-C_j\cos\left[\frac{(2j-1)\pi}{\tau_0}\frac{2x}{v}-\epsilon_j\right]\right\}.$$

A special case of interest in many applications follows from the foregoing by noting that in the limit as $x \to 0$ and $\epsilon_{g,j} \to 0$, $B_j \to 1$,

$$Q_{\tau oc2} = \frac{hA\tau_0t_g'}{2}\left[1+\frac{8}{\pi^2}\sum_{j=1}^{\infty}\frac{C_j\cos\epsilon_j}{(2j-1)^2}\right]$$

This represents an infinitely short regenerator, or a regenerator in which the amount of energy carried by the fluid streams is so large compared to the heat capacity of the solid that the attenuation of the stream temperature potential is negligible. The effectiveness of such regenerator is shown on Fig. 1. It indicates the extent to which the heat storage in the solid is dependent on the capability of the solid to absorb the energy supplied by an "inexhaustible" reservoir.

Résumé—Une solution analytique du problème du régénérateur unidirectionnel est présentée. L'efficacité d'un régénérateur à n'importe quelle distance arbitraire de l'entrée est définie en fonction de groupes sans dimensions qui régissent le problème, calculée pour une large gamme de paramètres et donnée à l'aide d'une série de graphiques. Bien qu'il tienne compte de l'hypothèse restrictive de l'égalité des capacités thermiques et de la constance des coefficients de transport de chaleur pour des écoulements de gaz chaud et froid, le calcul théorique diffère des méthodes classiques de calcul en ce que:

(1) Acune hypothèse simplificatrice n'a été faite au sujet des propriétés du matériau et des dimensions de la matrice poreuse, en dehoes de celle qui suppose que la conductivité thermique du solide est nulle dans le direction de l'écoulement gazeux.

(2) On n'a besoin d'aucune hypothèse sur la nature du profil longitudinal de température le long du régénérateur.

Les graphiques d'efficacité locale, qui décrivent le fonctionnement cyclique d'un régénérateur unidirectionnel, sont d'un type disponible jusqu'à présent seulement pour des récupérateurs ordinaires.

Zusammenfassung—Es wird eine analytische Lösung des Gleichstromregeneratorproblems gegeben. Die Wirksamkeit eines Regenerators in beliebigem Abstand vom Einlass ist in Form dimensionsloser, das Problem bestimmender Gruppen definiert, die für einen grossen Parameterbereich berechnet und in einer Reihe von Diagrammen angegeben sind. Während einschränkend gleiche Wärmekapazitäten und konstante Wärmeübergangskoeffizienten für den heissen und kalten Gasstrom beibehalten werden, unterscheidet sich die analytische Behandlung von den konventionellen Methoden darin, dass:

(1) kein vereinfachenden Annahmen hinsichtlich der Materialeigenschafter und Dimensionen der Wärmespeichermasse gemacht sind ausser der Forderung, dass die Wärmeleitfähigkeit in Richtung des Gasstroms Null ist;

(2) keine Annahmen über die Natur des Längstemperaturprofils im Regenerator notwendig sind.

Örtliche Wirksamkeitsdiagramme, welche die Arbeitsweise des Gleichstromregenerators bei zyklischer Beaufschlagung angeben, sind von der Art, wie sie bisher nur für gewöhnliche Rekuperatoren zur Verfügung standen.

Аннотация—Дано теоретическое решение задачи для прямоточного регенератора. Коэффициент регенерации на любом произвольном расстоянии от входа представлен безразмерными группами, определяющими задачу, которые рассчитаны для широкого диапазона параметров приводятся в ряде графиков. В то время как сохраняется условие равности теплоемкостей и постоянства коэффициентов теплообмена для потоков горячего и холодного газа, теоретический анализ отличается от обычных методов исследования тем, что:

(1) Не принимаются упрощающие предположения относительно свойств материала и размеров теплоаккумулирующей набивки за исключением нулевой теплопроводности твердого тела в направлении течения газа;

(2) Не требуется допущения о природе продольного профиля температуры вдоль теплообменника.

Графики локальных коэффициентов регенерации, описывающих циклическую работу прямоточного регенератора, принадлежат к типу графиков, имеющихся ло сих пор только для обычных рекуперативных теплообменников.